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A classification of accessible categories

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Dedicated to Max Kelly on the occasion of his 70th birthday

Abstract

For a suitable collection $\mathbb{D}$ of small categories, we define the $\mathbb{D}$-accessible categories, generalizing the λ-accessible categories of Lair, Makkai, and Paré; here the λ-accessible categories are seen as the $\mathbb{D}$-accessible categories where $\mathbb{D}$ consists of the λ-small categories. A small category $\mathscr{C}$ is called $\mathbb{D}$-filtered when $\mathscr{C}$-colimits commute with $\mathbb{D}$-limits in the category of sets. An object of a category is called $\mathbb{D}$-presentable when the corresponding representable functor preserves $\mathbb{D}$-filtered colimits. The $\mathbb{D}$-accessible categories are then the categories with $\mathbb{D}$-filtered colimits and a small set of $\mathbb{D}$-presentable objects which is "dense with respect to $\mathbb{D}$-filtered colimits".

We suppose always that $\mathbb{D}$ satisfies a technical condition called "soundness": this is the "suitable" case mentioned above. Every $\mathbb{D}$-accessible category is accessible; thus the choice of different sound $\mathbb{D}$ provides a classification of accessible categories, as referred to in the title. A surprising number of the main results from the theory of accessible categories remain valid in the $\mathbb{D}$-accessible context.

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The locally $\mathbb{D}$-presentable categories are defined as the cocomplete $\mathbb{D}$-accessible categories. When $\mathbb{D}$ consists of the finite categories, these are precisely the locally finitely presentable categories of Gabriel and Ulmer. When $\mathbb{D}$ consists of the finite discrete categories, these are the finitary varieties.

As a by-product of this theory, we prove that the free completion under $\mathbb{D}$-filtered colimits distributes over the free completion under limits. This result is new, even in the case where $\mathbb{D}$ is empty and $\mathbb{D}$-filtered colimits are just arbitrary (small) colimits.

MSC: 18C35; 18A30; 18A35; 18C30

Introduction

Accessible categories, as introduced by Lair [17], Makkai and Paré [20], form an important collection of categories because of their generality (they include all model categories of sketches and all categories of structures axiomatizable in first-order logic) and because a fruitful theory of accessibility has been developed. However, in that theory, somewhat complicated cardinality formulas emerge from time to time, concerning the question of which λ makes a given category λ-accessible. This has inspired us to consider the following question: can the theory of accessible categories be based on a choice of base-categories rather than a choice of cardinal numbers? In the present note we believe ourselves to have demonstrated that the answer is affirmative.

Recall [20] that a category $\mathscr{K}$ is called λ-accessible (for a regular cardinal number λ) provided that

(i) $\mathscr{K}$ has λ-filtered colimits;
(ii) $\mathscr{K}$ has a small set of λ-presentable objects whose one-step closure under λ-filtered colimits is all of $\mathscr{K}$.

A category is called accessible when it is λ-accessible for some λ. We refine this concept by choosing a small collection $\mathbb{D}$ of small categories (the "doctrine of $\mathbb{D}$-limits") and defining a small category $\mathscr{C}$ to be $\mathbb{D}$-filtered if $\mathscr{C}$-colimits commute in Set with limits of the doctrine; that is, limits of diagrams with domain in $\mathbb{D}$. Thus a category is filtered if it is FIN-filtered, where FIN is the doctrine of finite limits; more generally: a category is λ-filtered when it is $\mathbb{D}$-filtered for $\mathbb{D} = \lambda\text{-}\mathsf{LIM}$, the doctrine of λ-small limits, consisting of all categories of fewer than λ morphisms. The concept of $\mathbb{D}$-presentable object is then obvious: it is one whose Hom-functor preserves $\mathbb{D}$-filtered colimits. And a category $\mathscr{K}$ is called $\mathbb{D}$-accessible if it has $\mathbb{D}$-filtered colimits and a set of $\mathbb{D}$-presentable objects whose one-step closure under $\mathbb{D}$-filtered colimits is all of $\mathscr{K}$.

A special case has been studied in [6]: the doctrine $\mathbb{D} = \mathsf{FINPR}$ of finite products. There, FINPR-accessible categories are called generalized varieties. Among the complete (or cocomplete) categories, generalized varieties are precisely the varieties of (many sorted) finitary algebras. But there are other interesting examples: the categories of all fields or all linearly ordered sets are generalized varieties.

In fact we restrict ourselves to $\mathbb{D}$ which satisfy a technical condition called *soundness*, but this includes virtually all the interesting examples. For sound $\mathbb{D}$, we prove a number of results from the theory of accessible categories in the context of $\mathbb{D}$-accessible categories. Among other things, we show that every $\mathbb{D}$-accessible category is accessible, hence the title of our paper. For every small category $\mathscr{A}$, a free completion under $\mathbb{D}$-filtered colimits, denoted by $\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{A})$ (in honour of Grothendieck's concept $\mathsf{Ind}(\mathscr{A})$ in the case $\mathbb{D} = \mathsf{FIN}$) is proved to be $\mathbb{D}$-accessible; and conversely, every $\mathbb{D}$-accessible category is equivalent to $\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{A})$ for some small category $\mathscr{A}$.

What is the relation between $\mathbb{D}$-accessible categories and sketches? One of the most crucial results in the theory of accessible categories is that

$$\textit{accessible} = \textit{sketchable}.$$

There is no hope that an analogous result would be proved *within* one doctrine $\mathbb{D}$—in fact it does *not* hold for a fixed cardinal λ: there are $\aleph_0$-accessible categories that cannot be sketched by a finitary sketch, and finitary sketches whose categories of models are not $\aleph_0$-accessible; see [7]. However we do prove that

(i) every $\mathbb{D}$-accessible category can be sketched by a ($\mathbb{D}$-limit, colimit)-sketch;
(ii) if the colimit part is empty, the converse holds: $\mathbb{D}$-limit sketches have $\mathbb{D}$-accessible categories of models.

The $\mathbb{D}$-accessible categories in (ii) are, obviously, cocomplete; we call cocomplete $\mathbb{D}$-accessible categories *locally* $\mathbb{D}$-*presentable*, following the terminology of Gabriel and Ulmer [14]. Examples include: locally finitely presentable categories are precisely the case $\mathbb{D}=\mathsf{FIN}$, varieties are precisely the case $\mathbb{D}=\mathsf{FINPR}$. Another case of interest is the doctrine $\mathbb{D}=\mathsf{FINCL}$ of finite connected limits. We show that FINCL-accessible categories are precisely the categories $\mathsf{Fam}(\mathscr{K})$ where $\mathscr{K}$ is $\aleph_0$-accessible (and $\mathsf{Fam}(\mathscr{K})$ denotes the category of families, i.e., a free completion of $\mathscr{K}$ under coproducts). And the locally FINCL-presentable categories are the categories of the form $\mathsf{Fam}(\mathscr{K})$ for some $\aleph_0$-accessible category $\mathscr{K}$ with connected colimits.

We study the free completion $\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{A})$ beyond the need of $\mathbb{D}$-accessible categories: since not only small but in fact all categories have a free completion under $\mathbb{D}$-filtered colimits, we obtain a pseudomonad (in fact, a KZ-doctrine) $\mathbb{D}\text{-}\mathsf{Ind}$ on the quasi-category CAT of all categories. We prove that free $\mathbb{D}$-filtered colimits distribute over free limits, i.e., that $\mathbb{D}\text{-}\mathsf{Ind}$ has a distributive law over Lim (the pseudomonad of free completion under limits). In the doctrine $\mathbb{D}=\mathsf{FIN}$ this has been "partially proved" by Grothendieck and Verdier [9] and explicated in [2]. In the doctrine $\mathbb{D}=\mathsf{FINPR}$ this distributive law was established in [3], and our proof technique is the same.

A generalization of the concept of accessible category, obtained by considering an arbitrary class of colimits, has been studied by Hongde Hu [15]. Although this seems to be the same direction that our paper takes, the spirit is quite different: our aim is to refine the theory of accessible categories (by considering $\mathbb{D}$-accessibility for sound $\mathbb{D}$), not to generalize the concept.

Independently of our work, Pierre Ageron [8] has characterized the ($\mathbb{D}$-limit, colimit)-sketchable categories for certain large collections $\mathbb{D}$ of small categories. The most important case is where $\mathbb{D}$ consists of all connected categories; the resulting categories of models are called *normally accessible*. Our FINCL-accessible categories coincide with his finitely normally accessible ones. He also considers the case where $\mathbb{D}$ consists of all non-empty categories, in which case the resulting categories of models are called positively accessible.

We are very pleased to dedicate this paper to Max Kelly on the occasion of his 70th birthday; particularly so, since at a key point in the paper (Proposition 3.5) we use his characterization of the free completion of a category under a class of colimits. We are likewise pleased to acknowledge a number of helpful conversations with Max on topics closely related to this paper.

1. $\mathbb{D}$-filtered categories

Definition 1.1. We shall say that a collection $\mathbb{D}$ of small categories is a *limit doctrine*, or *doctrine* for short, if $\mathbb{D}$, seen as a full subcategory of Cat, is essentially small. A $\mathbb{D}$-limit is the limit of a functor with domain in $\mathbb{D}$. A category is called $\mathbb{D}$-*complete* if it has all $\mathbb{D}$-limits, and a functor between $\mathbb{D}$-complete categories is called $\mathbb{D}$-*continuous* if it preserves all $\mathbb{D}$-limits. Dually, there are the notions of $\mathbb{D}$-*cocompleteness* and $\mathbb{D}$-*cocontinuity*. We write $\mathbb{D}^{\mathsf{op}}$ for the doctrine consisting of all categories $\mathscr{D}^{\mathsf{op}}$ for $\mathscr{D} \in \mathbb{D}$.

Definition 1.2. A small category $\mathscr{C}$ is called $\mathbb{D}$-*filtered* if $\mathscr{C}$-colimits commute in Set with $\mathbb{D}$-limits.

Remark. Explicitly, this means that for every $\mathscr{D} \in \mathbb{D}$ and every functor $F : \mathscr{C} \times \mathscr{D} \to \mathsf{Set}$ the canonical map

$$\operatorname*{colim}_c \lim_d F(c,d) \to \lim_d \operatorname*{colim}_c F(c,d)$$

is an isomorphism. This is equivalent to asking that the functor colim $: [\mathscr{C}, \mathsf{Set}] \to \mathsf{Set}$ be $\mathbb{D}$-continuous, which is in turn equivalent to asking that the functor lim $: [\mathscr{D}, \mathsf{Set}] \to \mathsf{Set}$ be $\mathscr{C}$-cocontinuous for all $\mathscr{D} \in \mathbb{D}$.

Example 1.3. (i) The doctrine FIN of finite limits is the essentially small collection $\mathbb{D}$ of all finite categories. It has the well-known property that filtered colimits commute with FIN-limits and no other colimits do so—see Proposition 2.1 for the latter fact—thus

$$\text{FIN-filtered} \equiv \text{filtered}.$$

(ii) The doctrine FINPR of finite products is the essentially small collection of all finite discrete categories. A small category $\mathscr{C}$ is FINPR-filtered if and only if finite products commute with $\mathscr{C}$-colimits in Set. Such categories are called *sifted* in [1,6] (and for $\mathscr{C} \neq \emptyset$, *tamisante* in [18]). Every filtered small category is sifted, of course, but so is the scheme of reflexive coequalizers; that is, the category with two parallel arrows $f_0, f_1 : a \rightrightarrows b$ and a common section $d : a \leftarrow b$, so that $f_0 \circ d = \mathsf{id} = f_1 \circ d$. Thus sifted colimits encompass filtered colimits and reflexive coequalizers.

(iii) For every regular cardinal λ, denote by λ-LIM the doctrine of all λ-small limits: this is the essentially small collection of all categories with fewer than λ morphisms. Analogously to (i) above

$$\lambda\text{-Lim-filtered} \equiv \lambda\text{-filtered}.$$

(iv) For every regular cardinal λ denote by λ-PR the doctrine of all λ-small products: this is the essentially small collection of all discrete categories with fewer than λ objects. For $\lambda{=}\aleph_0$ we have λ-PR=FINPR, of course. For every regular cardinal $\lambda > \aleph_0$, it has been proved in [4] that whenever $\mathscr{C}$-colimits commute with λ-ary products in Set, then they commute with λ-ary limits. Thus in sharp contrast to (ii) above,

$$\text{if } \lambda > \aleph_0 \text{ then } \lambda\text{-PR-filtered} \equiv \lambda\text{-filtered}.$$

(v) For $\mathbb{D} = \emptyset$, the empty set, every small category is $\emptyset$-filtered.

(vi) Let FINCL denote the doctrine of finite connected limits; that is, the essentially small collection of all finite connected categories. Then FINCL-filtered categories are precisely the coproducts of filtered categories. In fact, since coproducts commute with connected limits in Set, every coproduct of filtered categories is FINCL-filtered. The converse follows from Proposition 2.1.

(vii) Let TERM denote the doctrine of the terminal object; that is, the doctrine consisting of the empty category. Then the TERM-filtered categories are the connected ones.

Definition 1.4. If $\mathscr{A}$ is a category with $\mathbb{D}$-filtered colimits, then an object A of $\mathscr{A}$ is said to be $\mathbb{D}$-*presentable* if the representable functor $\mathscr{A}(A,-) : \mathscr{A} \to \mathsf{Set}$ preserves $\mathbb{D}$-filtered colimits.

Example 1.5. (i) To be FIN-presentable is to be finitely presentable; to be λ-LIM-presentable is to be λ-presentable.

(ii) The FINPR-presentable objects were called *strongly finitely presentable* in [6] and *effective projective* in [24]. In a variety of algebras, an algebra is strongly finitely presentable if and only if it is regularly projective and finitely presentable (or, equivalently, it is a retract of a finitely generated free algebra).

(iii) An object is FINCL-presentable if and only if it is finitely presentable and *connected*, where A is called connected if $\mathscr{A}(A,-)$ preserves coproducts.

Lemma 1.6. *A* $\mathbb{D}^{\mathsf{op}}$*-colimit of* $\mathbb{D}$*-presentable objects is* $\mathbb{D}$*-presentable.*

Proof. Let $\mathscr{D}$ be in $\mathbb{D}$, and let $S : \mathscr{D}^{\mathsf{op}} \to \mathscr{K}$ be a diagram for which Sd is $\mathbb{D}$-representable for every object d in $\mathscr{D}$, and for which colim S exists. We show that for any diagram $H : \mathscr{J} \to \mathscr{K}$ for which $\mathscr{J}$ is $\mathbb{D}$-filtered, $\mathscr{K}(\operatorname{colim} S, -)$ preserves the colimit of H, as follows:

$$\mathscr{K}\left(\operatorname*{colim}_{d\in\mathscr{D}^{\mathrm{op}}} Sd, \operatorname{colim} H\right) \cong \lim_{d\in\mathscr{D}} \mathscr{K}(Sd, \operatorname{colim} H)$$

$$\cong \lim_{d\in\mathscr{D}} \operatorname*{colim}_{j\in\mathscr{J}} \mathscr{K}(Sd, Hj)$$

(since $\mathscr{K}(Sd,-)$ preserves $\mathbb{D}$-filtered colimits)

$$\cong \operatorname*{colim}_{j\in\mathscr{J}} \lim_{d\in\mathscr{D}} \mathscr{K}(Sd, Hj)$$

(since $\mathbb{D}$-limits commute with $\mathbb{D}$-filtered colimits)

$$\cong \operatorname*{colim}_{j\in\mathscr{J}} \mathscr{K}\left(\operatorname*{colim}_{d\in\mathscr{D}^{\mathrm{op}}} Sd, Hj\right). \qquad \square$$

Remark 1.7. More generally, a multicolimit of a $\mathbb{D}^{\mathrm{op}}$-diagram of $\mathbb{D}$-presentable objects has $\mathbb{D}$-presentable components.

Proof. Recall [11] that the multicolimit of a diagram $S:\mathscr{D}^{\mathrm{op}} \to \mathscr{K}$ is a collection of cocones

$$a_i : S \to \Delta A_i, \quad i \in I$$

such that for every cocone of $\mathbb{D}$ there exists a unique $i \in I$ for which that cocone factors through a_i and, moreover, the factorization is unique.

Suppose that each Sd is $\mathbb{D}$-presentable; we shall prove that the A_i are so for all $i \in I$. This goes as above, except that at the beginning we use

$$\coprod_{i\in I} \mathscr{K}(A_i, \operatorname{colim} H) \cong \lim_{d\in\mathscr{D}} \operatorname*{colim}_{j\in\mathscr{J}} \mathscr{K}(Sd, Hj)$$

and in the end we prove

$$\coprod_{i\in I} \mathscr{K}(A_i, \operatorname{colim} H) \cong \coprod_{i\in I} \operatorname*{colim}_{j\in\mathscr{J}} \mathscr{K}(A_i, H_j).$$

Then it remains only to observe that the last canonical isomorphism is "computed componentwise", in the sense that it is a coproduct of canonical isomorphisms

$$\mathscr{K}(A_i, \operatorname{colim} H) \cong \operatorname*{colim}_{j\in\mathscr{J}} \mathscr{K}(A_i, Hj) \quad (i \in I). \qquad \square$$

2. $\mathbb{D}$-flat functors

Recall that given a functor $F:\mathscr{A} \to \mathsf{Set}$, the category $\mathsf{Elts}(F)$ of elements of F has for objects the pairs (A, a) where A is an object of $\mathscr{A}$, and $a \in FA$; and for morphisms $f:(A,a) \to (B,b)$ those arrows $f:A \to B$ of $\mathscr{A}$ with $Ff(a) = b$.

Recall also that a functor $F:\mathscr{A}\to\mathsf{Set}$ with small domain is said to be *flat* if it satisfies one of the following equivalent conditions:

(i) F is a filtered colimit of representable functors;

(ii) $\mathsf{Elts}(F)^{\mathsf{op}}$ is filtered;

(iii) the Kan extension

$$\mathrm{Lan}_Y F:[\mathscr{A}^{\mathsf{op}},\mathsf{Set}]\to\mathsf{Set}$$

of F along the Yoneda embedding Y preserves finite limits.

If moreover $\mathscr{A}$ is finitely complete, these conditions are further equivalent to:

(iv) F preserves finite limits.

Part of the guiding philosophy of this project is that the equivalence of these notions is fundamental to the theory of accessible categories. It will therefore be crucial to have an analogous equivalence when we move from finite limits to $\mathbb{D}$-limits for a doctrine $\mathbb{D}$. The fact that this equivalence does *not* hold for arbitrary doctrines leads us to the most important definition of the paper, that of a *sound doctrine*. Before making the definition, we have the following:

Proposition 2.1. *Let $\mathscr{C}$ be a $\mathbb{D}$-filtered category and $\mathscr{D}\in\mathbb{D}$. Then the category of cocones of any functor $S:\mathscr{D}^{\mathsf{op}}\to\mathscr{C}$ is connected.*

Remark. The category of cocones has the expected choice of morphisms from $\alpha:S\to\Delta A$ to $\beta:S\to\Delta B$: those morphisms $f:A\to B$ of $\mathscr{C}$ with $f\circ\alpha=\beta$.

Proof. The category of cocones admits the following description. Let $\mathscr{C}(S,1):\mathscr{C}\to[\mathscr{D},\mathsf{Set}]$ be the evident functor taking an object c of $\mathscr{C}$ to the functor $\mathscr{C}(S-,c):\mathscr{D}\to\mathsf{Set}$. Then the category of cocones of S is the category of elements of the composite

$$\mathscr{C}\xrightarrow{\mathscr{C}(S,1)}[\mathscr{D},\mathsf{Set}]\xrightarrow{\lim}\mathsf{Set}.$$

Writing π_0 for the functor taking a category to its set of connected components, we have

$$\begin{aligned}\pi_0(\mathrm{cocone}(S)) &\cong \pi_0\left(\mathsf{Elts}\left(\lim_{d\in\mathscr{D}}\mathscr{C}(Sd,-)\right)\right)\\ &\cong \operatorname*{colim}_{c\in\mathscr{C}}\lim_{d\in\mathscr{D}}\mathscr{C}(Sd,c)\\ &\cong \lim_{d\in\mathscr{D}}\operatorname*{colim}_{c\in\mathscr{C}}\mathscr{C}(Sd,c)\\ &\cong \lim_{d\in\mathscr{D}}\Delta 1\\ &\cong 1. \qquad\square\end{aligned}$$

Definition 2.2. A doctrine $\mathbb{D}$ is said to be *sound* if the converse of Proposition 2.1 holds; that is, if a (small) category $\mathscr{C}$ is $\mathbb{D}$-filtered whenever the category of cocones of any functor $S : \mathscr{D}^{\mathsf{op}} \to \mathscr{C}$ with $\mathscr{D} \in \mathbb{D}$ is connected.

Example 2.3. (i) The doctrine FIN of finite limits is sound: if $\mathscr{C}$ is a small category in which finite diagrams have connected categories of cocones, then certainly they have non-empty categories of cocones; thus $\mathscr{C}$ is filtered. More generally, the doctrine λ-LIM of λ-small limits is sound.

(ii) The doctrine FINPR of finite products is sound. This non-trivial fact is due (in different language) to Foltz [12] and Lair [18]; a more detailed proof can be found in [6].

(iii) The doctrine FINCL of finite connected limits is sound. Let $\mathscr{C}$ be a small category for which every finite connected diagram has a connected category of cocones. Clearly the connected components of $\mathscr{C}$ must be filtered. Since a category is the coproduct of its connected components, $\mathscr{C}$ is a coproduct of filtered categories, so is FINCL-filtered. It follows that FINCL is sound.

(iv) The doctrine TERM of the terminal object is sound: the category of cocones over the empty diagram in $\mathscr{C}$ is just $\mathscr{C}$ itself, and we saw in Example 1.3 that $\mathscr{C}$ is TERM-filtered if and only if it is connected.

(v) The empty doctrine is sound.

(vi) The doctrine $\aleph_1$-PR of countable products is not sound. As seen in Example 1.3, the $\aleph_1$-PR-filtered categories are precisely the countably filtered ones. Let $\mathscr{C}$ be the free completion under countable coproducts of the category with two objects and two parallel non-identity arrows. Then for any countable discrete diagram in $\mathscr{C}$, the category of cocones has an initial object, so certainly is connected; but $\mathscr{C}$ is clearly not filtered. Similarly λ-PR is not sound if $\lambda > \aleph_0$.

(vii) The doctrine PB of pullbacks is not sound. Let G be a non-trivial group, seen as a one-object category. Clearly any $\mathbb{D}^{\mathsf{op}}$-diagram in G has a connected category of cocones; on the other hand G is not $\mathbb{D}$-filtered since the colimit functor $[G, \mathsf{Set}] \to \mathsf{Set}$ sends the pullback diagram on the left below to the diagram on the right, manifestly not a pullback:

$$\begin{tikzcd} G \times G \arrow[r] \arrow[d] & G \arrow[d] \\ G \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & 1. \end{tikzcd}$$

(viii) The union $\mathbb{D}$ of the doctrines PB and TERM is not sound—which is rather striking since $\mathbb{D}$ is clearly "equivalent" in some sense to the sound doctrine FIN. The $\mathbb{D}$-filtered categories are just the filtered ones, but if G is a non-trivial group, seen as a one-object category, then a diagram $S : \mathscr{D}^{\mathsf{op}} \to G$ with $\mathscr{D} \in \mathbb{D}$ clearly has a connected category of cocones, while G is certainly not filtered.

In the context of sound doctrines, we can prove the equivalence of the various possible definitions of $\mathbb{D}$-flat functors:

Theorem 2.4. *Let $\mathbb{D}$ be a sound doctrine. Then the following conditions on a functor $F:\mathscr{A}\to\mathsf{Set}$ with small domain are equivalent*:
(i) $\mathrm{Lan}_Y(F)$ *preserves $\mathbb{D}$-limits of representables*;
(ii) $\mathrm{Lan}_Y(F)$ *is $\mathbb{D}$-continuous*;
(iii) *F is a $\mathbb{D}$-filtered colimit of representables*;
(iv) $\mathsf{Elts}(F)^{\mathsf{op}}$ *is $\mathbb{D}$-filtered.*
If moreover $\mathscr{A}$ is $\mathbb{D}$-complete, these conditions are further equivalent to:
(v) *F is $\mathbb{D}$-continuous.*

Proof. (iv) $\Rightarrow$ (iii) is obvious, since every functor F is a canonical colimit of a diagram indexed by $\mathsf{Elts}(F)^{\mathsf{op}}$.

(iii) $\Rightarrow$ (ii). Let $\mathscr{C}$ be $\mathscr{D}$-filtered, and $S:\mathscr{C}\to\mathscr{A}^{\mathsf{op}}$ a functor for which the colimit of $YS:\mathscr{C}\to[\mathscr{A},\mathsf{Set}]$ is F. Then

$$\begin{aligned}\mathrm{Lan}_Y(F) &\cong \mathrm{Lan}_Y\left(\operatorname*{colim}_{c\in\mathscr{C}}\mathscr{A}(Sc,-)\right)\\ &\cong \operatorname*{colim}_{c\in\mathscr{C}}\mathrm{Lan}_Y(\mathscr{A}(Sc,-))\\ &\cong \operatorname*{colim}_{c\in\mathscr{C}}\mathsf{eval}_{Sc},\end{aligned}$$

where $\mathsf{eval}_a:[\mathscr{A},\mathsf{Set}]\to\mathsf{Set}$ is the functor given by evaluation at an object a of $\mathscr{A}$. Now such evaluation functors preserve arbitrary limits, so $\mathrm{Lan}_Y(F)$ will preserve whatever limits commute with $\mathscr{C}$-colimits, in particular the $\mathscr{D}$-limits.

(ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (iv). If $\mathrm{Lan}_Y F$ preserves $\mathscr{D}$-limits of representables, we prove that $\mathsf{Elts}(F)^{\mathsf{op}}$ is $\mathscr{D}$-filtered.

Thus, let $S:\mathscr{D}^{\mathsf{op}}\to\mathsf{Elts}(F)^{\mathsf{op}}$ be a diagram, with $\mathscr{D}\in\mathbb{D}$; we must prove that S has a connected category of cocones.

It is well-known that $F\cong(\mathrm{Lan}_Y F)\circ Y$, since Y is full and faithful. Denote by $U:\mathsf{Elts}(F)\to\mathscr{A}$ the natural forgetful functor. By assumption, $\mathrm{Lan}_Y F$ preserves the limit of

$$YUS^{\mathsf{op}}:\mathscr{D}\to[\mathscr{A}^{\mathsf{op}},\mathsf{Set}]$$

thus

$$\begin{aligned}\lim FUS^{\mathsf{op}} &\cong \lim(\mathrm{Lan}_Y(F)YUS^{\mathsf{op}})\\ &\cong \mathrm{Lan}_Y F(\lim YUS^{\mathsf{op}}).\end{aligned}$$

Now the limit of YUS^{op} in $[\mathscr{A}^{\mathsf{op}},\mathsf{Set}]$ can be described as the functor

$$H=\lim YUS^{\mathsf{op}}:\mathscr{A}^{\mathsf{op}}\to\mathsf{Set}$$

assigning to every object X of $\mathscr{A}$ the set HX of all cones of US^{op} with domain X in $\mathscr{A}$. Next $\mathrm{Lan}_Y F(H)$ is the colimit of the diagram HU^{op}; let

$$\alpha_{(A,a)} : HA \to \mathrm{Lan}_Y F(H), \quad (A,a) \in \mathsf{Elts}(F) \tag{1}$$

denote the colimit cocone. The above canonical isomorphism

$$\zeta : \mathrm{Lan}_Y F(H) \to \lim FUS^{\mathrm{op}}$$

is defined by

$$\zeta\alpha_{(A,a)} : (f_d : A \to US^{\mathrm{op}}d)_{d\in\mathscr{D}} \mapsto ((Ff_d)a)_{d\in\mathscr{D}},$$

where as usual we describe an element of the limit of FUS^{op} as a compatible $\mathscr{D}$-tuple in $\prod_{d\in\mathscr{D}} FUS^{\mathrm{op}}(d)$.

We now conclude that the category of cocones of S in $\mathsf{Elts}(F)^{\mathrm{op}}$ is connected. Put

$$S(d) = (B_d, b_d) \text{ for } b_d \in F(B_d), \quad d \in \mathsf{Obj}(\mathscr{D}^{\mathrm{op}}).$$

Then the collection (b_d) is an element of $\lim FUS^{\mathrm{op}}$. Thus there is some $(A,a) \in \mathsf{Elts}(F)$ and some cone $f_d : A \to US^{\mathrm{op}}(d)$ with $(Ff_d)a = b_d$ for all d. Consequently, S has a cocone $f_d : Sd \to (A,a)$ in $\mathsf{Elts}(F)^{\mathrm{op}}$. Given another cocone $f'_d : Sd \to (A',a')$, $d \in \mathsf{Obj}(\mathscr{D}^{\mathrm{op}})$, then the two elements $\zeta\alpha_{(A,a)}(f_d)$ and $\zeta\alpha_{(A',a')}(f'_d)$ of the colimit (1) coincide, so by the description of colimits in Set the two cocones are connected by a zig-zag in the category of all cocones of S.

(i) $\Leftrightarrow$ (v) is trivial in the case where $\mathscr{A}$ has $\mathbb{D}$-limits. □

Proposition 2.5. *If $\mathbb{D}$ is sound, then every small $\mathbb{D}^{\mathrm{op}}$-cocomplete category is $\mathbb{D}$-filtered.*

Proof. If $\mathscr{A}$ is $\mathbb{D}^{\mathrm{op}}$-cocomplete and $S : \mathscr{D}^{\mathrm{op}} \to \mathscr{A}$ with $\mathscr{D} \in \mathbb{D}$, then the category of cocones of S has an initial object, so certainly is connected. Thus $\mathscr{A}$ is $\mathbb{D}$-filtered, by soundness of $\mathbb{D}$. □

Remark 2.6. For sound $\mathbb{D}$, if $\mathscr{A}$ is a small category with $\mathbb{D}$-limits, then $F : \mathscr{A} \to \mathsf{Set}$ preserves $\mathbb{D}$-limits if and only if $\mathrm{Lan}_Y F$ does so. A remarkably large amount of the theory of $\mathbb{D}$-accessible categories developed below can be carried out for doctrines $\mathbb{D}$ satisfying this condition that "$\mathbb{D}$-continuity is equivalent to $\mathbb{D}$-flatness", which is strictly weaker than soundness. It is, however, convenient to have the characterization of $\mathbb{D}$-filtered categories afforded by the soundness condition; and the only doctrine we know which satisfies the weaker condition but is not sound is the somewhat bizarre doctrine consisting of the union of PB and TERM. We therefore choose sound doctrines as our basic objects of study.

Remark 2.7. The definition of sound doctrine also has a more abstract formulation: to say that a doctrine $\mathbb{D}$ is sound is equivalent to saying that for a functor $F : \mathscr{A} \to \mathsf{Set}$ with small domain, the left Kan extension $\mathrm{Lan}_Y F : [\mathscr{A}^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{Set}$ preserves $\mathbb{D}$-limits if and only if it preserves $\mathbb{D}$-limits of representables. That this condition holds for sound doctrines was proved in Theorem 2.4; for the converse, it suffices to consider F of the form $\Delta 1 : \mathscr{C}^{\mathsf{op}} \to \mathsf{Set}$.

If $\mathbb{D}$ is not sound, we are obliged to choose a particular characterization of flat functors as the basis for our notion of $\mathbb{D}$-flatness:

Definition 2.8. For a small category $\mathscr{A}$, a functor $F : \mathscr{A} \to \mathsf{Set}$ is called $\mathbb{D}$-*flat* if the left Kan extension

$$\mathrm{Lan}_Y F : [\mathscr{A}^{\mathsf{op}}, \mathsf{Set}] \to \mathsf{Set}$$

of F along the Yoneda embedding $Y : \mathscr{A} \to [\mathscr{A}^{\mathsf{op}}, \mathsf{Set}]$ preserves $\mathbb{D}$-limits.

Example 2.9. (i) The FIN-flat functors are precisely the flat ones. More generally, a functor is λ-LIM-flat if and only if it is λ-flat; or, equivalently, if it is a λ-filtered colimit of representable functors.

(ii) FINPR-flat functors have been called sifted-flat in [6], and it was proved there that they are precisely the sifted colimits of representable functors.

(iii) By Example 1.3, a functor $F : \mathscr{K}^{\mathsf{op}} \to \mathsf{Set}$ is FINCL-flat if and only if it is a coproduct of flat functors. Every FINCL-flat functor is evidently PB-flat, but in fact the reverse implication also holds: if $F : \mathscr{K}^{\mathsf{op}} \to \mathsf{Set}$ is PB-flat, then $\mathrm{Lan}_Y F : [\mathscr{K}, \mathsf{Set}] \to \mathsf{Set}$ preserves pullbacks; and so, by [10, Lemma 2.1], preserves all finite connected limits. But this means precisely that F is FINCL-flat.

Remark 2.10. For a sound doctrine $\mathbb{D}$ we can extend the concept of a $\mathbb{D}$-flat functor to large categories $\mathscr{A}$. Recall that a set-valued functor F is called *small* if it is a small colimit of representable functors, and thus has a left Kan extension $\mathrm{Lan}_Y(F)$ along the Yoneda embedding. We say that a functor $F : \mathscr{A} \to \mathsf{Set}$ is $\mathbb{D}$-*flat* if it is small and the Kan extension $\mathrm{Lan}_Y F$ is $\mathbb{D}$-continuous. If $\mathscr{A}$ is $\mathbb{D}$-complete, then $F : \mathscr{A} \to \mathsf{Set}$ is $\mathbb{D}$-flat if and only if it is small and $\mathbb{D}$-continuous.

3. $\mathbb{D}$-accessible categories

For the remainder of the paper we work only with sound doctrines.

Recall the concept of the free completion $\mathsf{Ind}(\mathscr{K})$ of a category $\mathscr{K}$ under filtered colimits, introduced by Grothendieck [9]. A concrete description of $\mathsf{Ind}(\mathscr{K})$ can be obtained by taking the category of all functors in $[\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ which are small, filtered colimits of representable functors. Furthermore, a functor $F : \mathscr{K}^{\mathsf{op}} \to \mathsf{Set}$ is a small, filtered colimit of representables if and only if it is FIN-flat in the sense of Remark 2.10; that is, if and only if it is small and $\mathrm{Lan}_Y F$ preserves finite limits. This can be generalized to an arbitrary sound doctrine $\mathbb{D}$ as follows.

Notation 3.1. Let $\mathbb{D}$ be a sound doctrine. For every category $\mathscr{K}$ we denote by $\mathbb{D}$-$\mathsf{Ind}(\mathscr{K})$ the category of all $\mathbb{D}$-flat functors in $[\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ and by $\eta_{\mathscr{K}} : \mathscr{K} \to \mathbb{D}$-$\mathsf{Ind}(\mathscr{K})$ the codomain restriction of the Yoneda embedding.

Example 3.2. (i) FIN-Ind is just Grothendieck's Ind.

(ii) FINPR-Ind is called Sind in [6] where it is proved that $\mathsf{Sind}(\mathscr{K})$ is the free completion of $\mathscr{K}$ under sifted colimits.

(iii) FINCL-$\mathsf{Ind}(\mathscr{K})$ is the free completion of $\mathsf{Ind}(\mathscr{K})$ under coproducts. That is, if we denote by Fam the usual category of small families (the free coproduct completion), then

$$\mathsf{FINCL}\text{-}\mathsf{Ind}(\mathscr{K}) \simeq \mathsf{Fam}(\mathsf{Ind}(\mathscr{K})).$$

We saw in Example 2.9 that a functor is FINCL-flat if and only if it is a coproduct of flat functors; since coproducts in $[\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ are disjoint and universal, it follows that natural transformations preserve coproduct-components, and so that the above equivalence holds.

(iv) For $\mathbb{D} = \emptyset$, since every category is $\emptyset$-filtered, a functor $F : \mathscr{K}^{\mathsf{op}} \to \mathsf{Set}$ is $\emptyset$-flat iff it is *small*, i.e., a small colimit of representable functors. The category

$$\mathsf{Colim}(\mathscr{K}) =_{\mathsf{def}} \emptyset\text{-}\mathsf{Ind}(\mathscr{K})$$

of all small functors in $[\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ is well-known to be the free completion of $\mathscr{K}$ under colimits; see [19].

By a free completion of $\mathscr{K}$ under $\mathbb{D}$-filtered colimits we mean, of course, a functor

$$\eta_{\mathscr{K}} : \mathscr{K} \to \mathscr{K}^*$$

for which

(i) $\mathscr{K}^*$ has $\mathbb{D}$-filtered colimits;

(ii) given a category $\mathscr{L}$ with $\mathbb{D}$-filtered colimits, then for every functor $F : \mathscr{K} \to \mathscr{L}$ there is a functor $F^* : \mathscr{K}^* \to \mathscr{L}$, unique up to natural isomorphism, such that $F \cong F^* \circ \eta_{\mathscr{K}}$, and F^* preserves $\mathbb{D}$-filtered colimits.

Recall [25] that (ii) implies that $F \mapsto F^*$ yields an equivalence of the categories $[\mathscr{K}, \mathscr{L}]$ and $[\mathscr{K}^*, \mathscr{L}]^{\mathbb{D}}$ (the full subcategory of $[\mathscr{K}^*, \mathscr{L}]$ consisting of all functors preserving $\mathbb{D}$-filtered colimits), pseudoinverse to composition with $\eta_{\mathscr{K}}$.

Proposition 3.3. *Let $\mathbb{D}$ be a sound doctrine. For every category $\mathscr{K}$*

$$\eta_{\mathscr{K}} : \mathscr{K} \to \mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K})$$

is the free completion under $\mathbb{D}$-filtered colimits of $\mathscr{K}$.

Proof. The desired free completion is the closure in $[\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ of the representables under $\mathbb{D}$-filtered colimits—see [19] for a more general result. Since we saw in Theorem 2.4 and Remark 2.10 that every $\mathbb{D}$-flat functor is a $\mathbb{D}$-filtered colimit of representables, it will suffice to verify that the $\mathbb{D}$-flat functors are closed in $[\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ under $\mathbb{D}$-filtered colimits. But this is easy: if the small category $\mathscr{C}$ is $\mathbb{D}$-filtered, and $S : \mathscr{C} \to [\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ is a diagram of $\mathbb{D}$-flat functors then $\mathrm{Lan}_Y(\mathrm{colim}_{C \in \mathscr{C}} SC) \cong \mathrm{colim}_{C \in \mathscr{C}} \mathrm{Lan}_Y(SC)$. Each

$\mathrm{Lan}_Y(SC)$ preserves $\mathbb{D}$-limits, and $\mathscr{C}$-colimits commute with $\mathbb{D}$-limits, so $\mathrm{Lan}_Y(\operatorname{colim} S)$ preserves $\mathbb{D}$-limits. □

Recall that a category $\mathscr{K}$ is said to be *finitely accessible* if it has filtered colimits and a small set $\mathscr{A}$ of finitely presentable objects such that every object is a filtered colimit of objects from $\mathscr{A}$. Recall further that $\mathscr{K}$ is finitely accessible if and only if it has the form $\mathsf{Ind}(\mathscr{A})$ for some small category $\mathscr{A}$; that is, if and only if $\mathscr{K}$ is the free completion under filtered colimits of a small category. We can extend these ideas to give a notion of $\mathbb{D}$-accessibility for an arbitrary sound doctrine $\mathbb{D}$.

Definition 3.4. For a sound doctrine $\mathbb{D}$, a category $\mathscr{K}$ is said to be $\mathbb{D}$-accessible if it has $\mathbb{D}$-filtered colimits and a small set $\mathscr{A}$ of $\mathbb{D}$-presentable objects such that every object is a $\mathbb{D}$-filtered colimit of objects from $\mathscr{A}$.

Proposition 3.5. *For a category $\mathscr{K}$, the following are equivalent*:
(i) *$\mathscr{K}$ is $\mathbb{D}$-accessible*;
(ii) *$\mathscr{K}$ is equivalent to $\mathbb{D}$-$\mathsf{Ind}(\mathscr{A})$ for a small category $\mathscr{A}$*;
(iii) *$\mathscr{K}$ is the free completion under $\mathbb{D}$-filtered colimits of a small category $\mathscr{A}$*.

Proof. (ii) $\Leftrightarrow$ (iii) is immediate from Proposition 3.3.

(i) $\Rightarrow$ (iii) follows from [16, Proposition 5.62], which includes in particular the fact that $\mathscr{K}$ is the free completion under $\mathbb{D}$-filtered colimits of a category $\mathscr{A}$ if and only if:

- $\mathscr{K}$ has $\mathbb{D}$-filtered colimits;
- there is a fully faithful functor $J : \mathscr{A} \to \mathscr{K}$;
- the closure in $\mathscr{K}$ under $\mathbb{D}$-filtered colimits of the image of J is $\mathscr{K}$ itself;
- for each object A of $\mathscr{A}$ the functor $\mathscr{K}(JA, -)$ preserves $\mathbb{D}$-filtered colimits.

These conditions follow immediately from the definition of $\mathbb{D}$-accessibility.

(iii) $\Rightarrow$ (i). We know by [16, Proposition 5.62] that $\mathscr{K}$ has $\mathbb{D}$-filtered colimits and there is a set $\mathscr{A}$ of $\mathbb{D}$-presentable colimits whose closure in $\mathscr{K}$ under $\mathbb{D}$-filtered colimits is $\mathscr{K}$ itself. It remains to show that this closure can be formed in one step; that is, every object of $\mathscr{K}$ is a $\mathbb{D}$-filtered colimit of objects in $\mathscr{A}$. To see this, observe that $\mathscr{K} \simeq \mathbb{D}$-$\mathsf{Ind}(\mathscr{A})$, and that every $\mathbb{D}$-flat functor is a $\mathbb{D}$-filtered colimit of representables, by Theorem 2.4. □

Remark 3.6. It is clear that the definition of $\mathbb{D}$-accessible category makes perfectly good sense for an arbitrary doctrine $\mathbb{D}$, but we have been unable in that generality to prove that every $\mathbb{D}$-accessible category is accessible, and it is for this reason that we have chosen to define $\mathbb{D}$-accessibility only for sound $\mathbb{D}$.

Example 3.7. (i) The FIN-accessible categories are the finitely accessible (or $\aleph_0$-accessible) categories of Lair [17] and Makkai-Paré [20]. More generally, λ-LIM-accessibility is just λ-accessibility.

(ii) FINPR-accessible categories were called generalized varieties in [6]. Among cocomplete categories, these are precisely the many-sorted finitary varieties.

(iii) $\emptyset$-accessible categories are precisely the presheaf categories $[\mathscr{A}^{\mathsf{op}}, \mathsf{Set}]$ for small $\mathscr{A}$.

(iv) FINCL-accessible categories are precisely the categories $\mathsf{Fam}(\mathscr{K})$ with $\mathscr{K}$ finitely accessible; see 3.2.

Notation 3.8. For every category $\mathscr{K}$ we denote by $\mathscr{K}_{\mathbb{D}}$ the full subcategory of $\mathbb{D}$-presentable objects.

Lemma 3.9. If $\mathscr{K}$ is a $\mathbb{D}$-accessible category, then
(i) $\mathscr{K}_{\mathbb{D}}$ is dense in $\mathscr{K}$;
(ii) $\mathscr{K}_{\mathbb{D}}$ is essentially small, and consists of the retracts of objects of $\mathscr{A}$ (as in Definition 3.4);
(iii) if K is an object of $\mathscr{K}$ then the comma-category $\mathscr{K}_{\mathbb{D}} \downarrow K$ is $\mathbb{D}$-filtered.

Proof. (i) We know that every object of $\mathscr{K}$ is a ($\mathbb{D}$-filtered) colimit of objects $A \in \mathscr{A}$, thus of objects $A \in \mathscr{K}_{\mathbb{D}}$, and that the representable functor $\mathscr{K}(K, -)$ preserves this colimit if $K \in \mathscr{K}_{\mathbb{D}}$. Thus $\mathscr{K}_{\mathbb{D}}$ is dense.

(ii) Given $K \in \mathscr{K}_{\mathbb{D}}$ expressed as the colimit of a $\mathbb{D}$-filtered diagram $S : \mathscr{J} \to \mathscr{A}$, since $\mathscr{K}(K, -)$ preserves colim S, we know that id_K factors through one of the colimit maps $k_d : Sd \to K$; thus $k_d \circ e = \mathsf{id}$ for some $d \in \mathscr{D}$ and some $e : K \to Sd$. Thus K is a retract of an object in $\mathscr{A}$. Since $\mathscr{K}$ has a colimit-dense small set, it is co-well-powered with respect to retracts. Therefore, $\mathscr{K}_{\mathbb{D}}$ is essentially small.

(iii) To prove that $\mathscr{K}_{\mathbb{D}} \downarrow K$ is $\mathbb{D}$-filtered, it will suffice to find a $\mathbb{D}$-filtered category $\mathscr{J}$ with a final functor $H : \mathscr{J} \to \mathscr{K}_{\mathbb{D}} \downarrow K$; that is , a functor H for which if $\varepsilon : Q \to K$ is in $\mathscr{K}_{\mathbb{D}} \downarrow K$, then the comma category $(Q, \varepsilon) \downarrow H$ is connected.

To do this, express K as the colimit (in $\mathscr{K}$) of a diagram $S : \mathscr{J} \to \mathscr{A}$, where $\mathscr{J}$ is $\mathbb{D}$-filtered. The diagram S along with its colimit cocone provides the desired functor $H : \mathscr{J} \to \mathscr{K}_{\mathbb{D}} \downarrow K$; the fact that $(Q, \varepsilon) \downarrow H$ is connected follows from the fact that Q is $\mathbb{D}$-presentable. $\square$

4. $\mathbb{D}$-Sketches

Recall that all doctrines considered are assumed to be sound.

One of the most fundamental results on accessible categories is that they coincide with categories of models of sketches; see [17,20]. However this does not hold in a given doctrine, such as λ-LIM: it is not true that λ-accessibility and sketchability by λ-small sketches coincide: see [7]. One implication does hold, and it can be generalized to all sound doctrines.

Recall [5] that a *sketch* is a quadruple $\mathscr{S} = (\mathscr{A}, \mathbb{L}, \mathbb{C}, \sigma)$ consisting of a small category $\mathscr{A}$, sets $\mathbb{L}$ and $\mathbb{C}$ of (small) diagrams in $\mathscr{A}$, and a map σ assigning a cone to every diagram in $\mathbb{L}$ and a cocone to every diagram in $\mathbb{C}$. *Models* of $\mathscr{S}$ are functors in $[\mathscr{A}, \mathsf{Set}]$ mapping $\sigma(D)$ to a limit cone if $D \in \mathbb{L}$ and to a colimit cocone if $D \in \mathbb{C}$. The full subcategory $\mathsf{Mod}(\mathscr{S}) \subseteq [\mathscr{A}, \mathsf{Set}]$ of all models is accessible, and every accessible category is equivalent to one of the form $\mathsf{Mod}(\mathscr{S})$; see [17,20]. We say that a category is *sketchable* by $\mathscr{S}$ if it is equivalent to $\mathsf{Mod}(\mathscr{S})$.

Definition 4.1. A sketch is called a $\mathbb{D}$-*sketch* if every diagram in $\mathbb{L}$ has scheme in $\mathbb{D}$ (no restriction is made on the diagrams in $\mathbb{C}$).

Proposition 4.2. *Every* $\mathbb{D}$-*accessible category is sketchable by a* $\mathbb{D}$-*sketch.*

Proof. It is sufficient to sketch $\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{A})$ for every small category $\mathscr{A}$. Let $\mathscr{B}$ denote the full subcategory of $[\mathscr{A}^{\mathsf{op}}, \mathsf{Set}]$ consisting of the representables and the $\mathbb{D}$-limits of representables. Let $\mathbb{L}$ consist of all diagrams $S : \mathscr{D} \to \mathscr{B}$ with $\mathscr{D} \in \mathbb{D}$ which land among the representables, and let σ assign to each such diagram its limit cone. Let $\mathbb{C}$ contain, for each F in $\mathscr{B}$, the canonical diagram $\mathscr{A} \downarrow F \to \mathscr{B}$, and let σ assign to each such diagram its colimit cocone. If $\mathscr{S}$ is the sketch $(\mathscr{B}, \mathbb{L}, \mathbb{C}, \sigma)$, we claim that $\mathsf{Mod}(\mathscr{S}) \simeq \mathbb{D}\text{-}\mathsf{Ind}(\mathscr{A})$. In fact for every object $F : \mathscr{A}^{\mathsf{op}} \to \mathsf{Set}$ of $\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{A})$, since $\mathrm{Lan}_Y F$ preserves $\mathbb{D}$-limits and colimits, the domain-restriction of $\mathrm{Lan}_Y F$ to $\mathscr{B}$ is a model $\hat{F} \in \mathsf{Mod}(\mathscr{S})$. It is clear that this defines a full and faithful functor

$$\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{A}) \to \mathsf{Mod}(\mathscr{S}), \quad F \mapsto \hat{F}.$$

Conversely, given a model $G : \mathscr{B} \to \mathsf{Set}$ of $\mathscr{S}$, let $F : \mathscr{A} \to \mathsf{Set}$ denote the domain restriction of G to $\mathscr{A}$. Due to the colimits in $\mathscr{S}$ we see that $G \cong \mathrm{Lan}_Y F|_{\mathscr{B}}$. Consequently, $G \cong \hat{F}$. $\square$

Remark 4.3. A $\mathbb{D}$-sketchable category need not be $\mathbb{D}$-accessible: see [6] for a counterexample with $\mathbb{D} = \mathsf{FINPR}$ and [5] for a counterexample with $\mathbb{D} = \mathsf{FIN}$.

Corollary 4.4. *Every* $\mathbb{D}$-*accessible category is accessible.*

Proof. In fact, every sketchable category is accessible; see [20]. $\square$

If λ and μ are regular cardinals and $\lambda < \mu$, then certainly every category with λ-filtered colimits must have μ-filtered colimits, but it may not be the case that every λ-accessible category is μ-accessible. Thus we cannot hope that if $\mathbb{D} \subseteq \lambda\text{-}\mathsf{LIM}$ then every $\mathbb{D}$-accessible category is λ-accessible, but we do have:

Theorem 4.5. *Let* $\mathbb{D}$ *be a sound doctrine with* $\mathbb{D} \subseteq \lambda\text{-}\mathsf{LIM}$, *and let* μ *be a regular cardinal for which* (i) *every* λ-*accessible category is* μ-*accessible*, *and* (ii) *if* $\alpha < \lambda$ *then* $\alpha^\alpha < \mu$. *Then every* $\mathbb{D}$-*accessible category is* μ-*accessible.*

Proof. (*I*) Let us prove that for every $\mathbb{D}$-filtered category $\mathscr{C}$, the collection of all μ-small $\mathbb{D}$-filtered subcategories (not necessarily full) of $\mathscr{C}$ is μ-filtered. To do so, it will suffice to show that every μ-small subcategory $\mathscr{B}$ of $\mathscr{C}$ is contained in a

μ-small $\mathbb{D}$-filtered subcategory $\mathscr{B}'$. We define $\mathscr{B}' = \bigcup_{i<\lambda} \mathscr{B}_i$ by the following transfinite induction, in which each $\mathscr{B}_i$ is μ-small.

First step: $\mathscr{B}_0 = \mathscr{B}$.

Isolated step: Given $\mathscr{B}_i$, let $\mathscr{B}_{i+1}$ be the compositional closure of the following extension of $\mathscr{B}_i$. For every diagram $S : \mathscr{D} \to \mathscr{B}_i$ with $\mathscr{D} \in \mathbb{D}$ we add to $\mathscr{B}_i$

(i) a cocone of S in $\mathscr{C}$, in case S has no cocone in $\mathscr{B}_i$;
(ii) a zigzag in $\mathscr{C}$ connecting any non-connected pair of cocones of S.

We claim that this can be done in such a way that $\mathscr{B}_{i+1}$ has fewer than μ morphisms. To this end, we shall prove that there are fewer than μ representative diagrams $S : \mathscr{D} \to \mathscr{B}_i$ with $\mathscr{D} \in \mathbb{D}$; then for each diagram S we can merely add fewer than λ morphisms in case (i) and fewer than $\lambda\mu^2 = \mu$ morphisms in case (ii). Since $\mathbb{D} \subseteq \lambda$-$\mathsf{LIM}$, we can assume that $\mathbb{D}$ has cardinality at most λ. Thus it will suffice to show that for a given $\mathscr{D} \in \mathbb{D}$ the number of all diagrams $S : \mathscr{D} \to \mathscr{B}_i$ is less than μ. Since every λ-accessible category is μ-accessible, by [6, Theorem 2.11] the set $P_\lambda(\operatorname{mor} \mathscr{B}_i)$ of all λ-small sets of morphisms of $\mathscr{B}_i$ has a final set, say $\mathscr{R}$, of cardinality less than μ. Given an element $\mathscr{A}$ of $\mathscr{R}$, the number of all diagrams $\mathscr{D} \to \mathscr{B}_i$ which only use morphisms in $\mathscr{A}$ is at most $\operatorname{card} \mathscr{A}^{\operatorname{card} \mathscr{D}}$. Now if $\alpha = \max(\operatorname{card} \mathscr{A}, \operatorname{card} \mathscr{D})$, then $\alpha < \lambda$, so $\alpha^\alpha < \mu$. It follows that the number of diagrams $\mathscr{D} \to \mathscr{B}_i$ which use only morphisms in A is less than μ.

Limit step: $\mathscr{B}_i = \bigcup_{j<i} \mathscr{B}_j$ if i is a limit ordinal.

It is now clear that $\mathscr{B}'$ has fewer than μ morphisms and is $\mathbb{D}$-filtered.

(*II*) Every $\mathbb{D}$-accessible category $\mathscr{K}$ has λ-filtered colimits, and so in particular has μ-filtered colimits. Let $\mathscr{A}$ be a set of representatives of the $\mathbb{D}$-presentable objects. Then each object in $\mathscr{A}$ is μ-presentable. Form a set $\bar{\mathscr{A}}$ of representatives of the μ-small $\mathbb{D}$-filtered colimits of objects in $\mathscr{A}$. Then every object of $\bar{\mathscr{A}}$ is μ-presentable; we shall show that every object of $\mathscr{K}$ is a μ-filtered colimit of objects in $\bar{\mathscr{A}}$. Given an object K of $\mathscr{K}$, write K as the colimit of a diagram $S : \mathscr{J} \to \mathscr{A}$ where $\mathscr{J}$ is $\mathbb{D}$-filtered. For each μ-small $\mathbb{D}$-filtered subcategory $\mathscr{J}_t$ of $\mathscr{J}$ we can form the colimit K_t of the restriction of S to $\mathscr{J}_t$, and then K_t will be (isomorphic to something) in $\bar{\mathscr{A}}$, by construction of $\bar{\mathscr{A}}$. Finally K is the colimit of the K_t, and by (*I*) this colimit is μ-filtered. □

Corollary 4.6. *If* $\mathbb{D} \subseteq \mathsf{FIN}$ *then every* $\mathbb{D}$*-accessible category is* $\aleph_1$*-accessible.*

Open problem 4.7. *If* $\mathbb{D} \subseteq \mathsf{FIN}$ *is every* $\mathbb{D}$*-accessible category finitely accessible*? *This is not known even if* $\mathbb{D} = \mathsf{FINPR}$.

Remark 4.8. This question has been answered in the affirmative by Ageron [8] in two cases: where $\mathbb{D}$ is FINCL and where $\mathbb{D}$ consists of the finite non-empty categories. It obviously also true if $\mathbb{D}$ is empty.

5. Locally $\mathbb{D}$-presentable categories

We continue to suppose that all doctrines considered are sound.

Definition 5.1. *A category is called* locally $\mathbb{D}$-presentable *if it is cocomplete and* $\mathbb{D}$*-accessible*.

Remark 5.2. A $\mathbb{D}$-accessible category is complete if and only if it is cocomplete, since this is true for accessible categories [20] and every $\mathbb{D}$-accessible category is accessible.

Example 5.3. (i) The locally FIN-presentable categories are precisely the locally finitely presentable categories of Gabriel and Ulmer [14]. More generally, the locally λ-LIM-presentable categories are the locally λ-presentable categories.

(ii) The locally FINPR-presentable categories are precisely the finitary varieties; see [6].

(iii) Locally FINCL-presentable categories are precisely the categories $\mathsf{Fam}(\mathscr{K})$ where $\mathscr{K}$ is $\aleph_0$-accessible and has connected colimits; see Example 3.7.

(iv) Locally $\emptyset$-presentable categories are precisely the presheaf categories $[\mathscr{A}^{\mathsf{op}}, \mathsf{Set}]$ with $\mathscr{A}$ small; see Example 3.7.

Notation 5.4. *Let $\mathscr{A}$ be a small $\mathbb{D}$-complete category. The full subcategory of* $[\mathscr{A}, \mathsf{Set}]$ *formed by all $\mathbb{D}$-continuous functors is denoted by* $\mathbb{D}$-$\mathsf{Cont}(\mathscr{A})$.

Theorem 5.5. *Let $\mathbb{D}$ be a sound doctrine. The following conditions on a category $\mathscr{K}$ are equivalent*:

(i) *$\mathscr{K}$ is locally $\mathbb{D}$-presentable*;

(ii) *$\mathscr{K}$ is a free completion of a small $\mathbb{D}^{\mathsf{op}}$-cocomplete category under $\mathbb{D}$-filtered colimits*;

(iii) *$\mathscr{K}$ is sketchable by a limit $\mathbb{D}$-sketch*;

(iv) *$\mathscr{K}$ is equivalent to* $\mathbb{D}$-$\mathsf{Cont}(\mathscr{A})$ *for a small $\mathbb{D}$-complete category $\mathscr{A}$*.

Proof. (iv) $\Rightarrow$ (i). The category $\mathbb{D}$-$\mathsf{Cont}(\mathscr{A}) = \mathbb{D}$-$\mathsf{Ind}(\mathscr{A}^{\mathsf{op}})$ is $\mathbb{D}$-accessible by Proposition 3.5; and cocomplete, since it is reflective in $[\mathscr{A}, \mathsf{Set}]$.

(i) $\Rightarrow$ (ii). If $\mathscr{K}$ is locally $\mathbb{D}$-presentable then it is the free completion of $\mathscr{K}_{\mathbb{D}}$ under $\mathbb{D}$-filtered colimits, and $\mathscr{K}_{\mathbb{D}}$ has $\mathbb{D}^{\mathsf{op}}$-colimits by Lemma 1.6.

(ii) $\Rightarrow$ (iii). Consider the limit $\mathbb{D}$-sketch on the opposite of the given $\mathbb{D}^{\mathsf{op}}$-cocomplete category with $\mathbb{L} =$ all $\mathbb{D}$-diagrams and $\sigma =$ limit cones.

(iii) $\Rightarrow$ (iv). If $\mathscr{K} \simeq \mathsf{Mod}(\mathscr{S})$ for a limit $\mathbb{D}$-sketch $\mathscr{S}$ on a category $\mathscr{A}$, write $Z : \mathscr{A} \to \mathsf{Mod}(\mathscr{S})^{\mathsf{op}}$ for the fully faithful functor sending an object A to $\mathscr{A}(A, -)$. Let $\mathscr{A}^*$ be the closure under $\mathbb{D}$-limits of the image. Then $\mathsf{Mod}(\mathscr{S}) \simeq \mathbb{D}$-$\mathsf{Cont}(\mathscr{A}^*)$. $\square$

Definition 5.6. Let $\mathscr{K}$ be a category with $\mathbb{D}$-filtered colimits. A $\mathbb{D}$-orthogonality class of $\mathscr{K}$ is a full subcategory consisting of all objects orthogonal to a small set $\mathscr{M}$ of morphisms in $\mathscr{K}$ with $\mathbb{D}$-presentable domains and codomains. We write $\mathscr{M}^{\perp}$ for the full subcategory.

Proposition 5.7. *The locally $\mathbb{D}$-presentable categories are precisely the categories equivalent to $\mathbb{D}$-orthogonality classes of presheaf categories. Moreover, any $\mathbb{D}$-orthogonality class of a locally $\mathbb{D}$-presentable category is itself locally $\mathbb{D}$-presentable.*

Proof. Let $\mathscr{K}$ be locally $\mathbb{D}$-presentable . Then $\mathscr{K} \simeq \mathbb{D}\text{-}\mathsf{Cont}(\mathscr{A})$ for a small $\mathbb{D}$-complete category $\mathscr{A}$. For each diagram $S:\mathscr{D} \to \mathscr{A}$ with $\mathscr{D} \in \mathbb{D}$ there is a canonical morphism

$$m_S : \operatorname*{colim}_{d\in\mathscr{D}^{\mathsf{op}}} \mathscr{A}(Sd,-) \to \mathscr{A}\left(\lim_{d\in\mathscr{D}} Sd, -\right)$$

in $[\mathscr{A},\mathsf{Set}]$ and one easily verifies that $F:\mathscr{A} \to \mathsf{Set}$ preserves the limit of S if and only if it is orthogonal to m_S. Thus $\mathbb{D}\text{-}\mathsf{Cont}(\mathscr{A}) = \mathscr{M}^{\perp}$ where $\mathscr{M}$ consists of the m_S for all $S:\mathscr{D} \to \mathscr{A}$ with $\mathscr{D} \in \mathbb{D}$. We must verify that the m_S have $\mathbb{D}$-presentable domains and codomains; but the codomain of m_S is representable, and so $\mathbb{D}$-presentable for *any* $\mathbb{D}$; while the domain is a $\mathbb{D}^{\mathsf{op}}$-colimit of representables, so this follows from Lemma 1.6.

To prove the converse, we verify the second part of the proposition: if $\mathscr{L}$ is locally $\mathbb{D}$-presentable, then so is $\mathscr{M}^{\perp}$ for any set $\mathscr{M}$ of morphisms of $\mathscr{L}$ with $\mathbb{D}$-presentable domains and codomains. Since $\mathscr{M}^{\perp}$ is closed under limits in $\mathscr{L}$, and $\mathscr{L}$ is complete (see Remark 5.2), it is sufficient to prove that $\mathscr{M}^{\perp}$ is $\mathbb{D}$-accessible.

$\mathscr{M}^{\perp}$ is obviously closed under $\mathbb{D}$-filtered colimits in $\mathscr{L}$. Let λ be a cardinal such that each $\mathscr{D} \in \mathbb{D}$ is λ-small; then every λ-filtered category is $\mathbb{D}$-filtered. Thus $\mathscr{M}^{\perp}$ is closed in $\mathscr{L}$ under λ-filtered colimits, hence reflective in $\mathscr{L}$ by [5, 1.66]. Let $\mathscr{A}$ be a set as in Definition 3.4 and let $\mathscr{A}^*$ be the set of all reflections into $\mathscr{M}^{\perp}$ of objects of $\mathscr{A}$. A reflection of a $\mathbb{D}$-presentable object of $\mathscr{K}$ is clearly $\mathbb{D}$-presentable in $\mathscr{M}^{\perp}$, and a $\mathbb{D}$-filtered colimit of $\mathscr{A}$-objects is reflected onto a $\mathbb{D}$-filtered colimit of $\mathscr{A}^*$-objects in $\mathscr{M}^{\perp}$. Therefore, $\mathscr{M}^{\perp}$ is $\mathbb{D}$-accessible. $\square$

The above results generalize immediately to categories having multicolimits but not colimits in general (see Remark 1.7). Diers has introduced *locally multipresentable categories* as accessible categories with multicolimits [11]. He proved that they are precisely the accessible categories with connected limits, or equivalently those which can be sketched by a *limit-coproduct* sketch (i.e., a sketch for which all diagrams of $\mathbb{C}$ are discrete).

Definition 5.8. *A category is called* locally $\mathbb{D}$-multipresentable *if it is* $\mathbb{D}$*-accessible and has multicolimits.*

Remark. Before generalizing the above description of locally $\mathbb{D}$-presentable categories as categories of $\mathbb{D}$-continuous functors, we should explain what we mean when we say that a functor $F:\mathscr{B} \to \mathsf{Set}$ is $\mathbb{D}$-multicontinuous. If $\mathscr{A}$ has $\mathbb{D}$-multicolimits, we say that $F:\mathscr{A}^{\mathsf{op}} \to \mathsf{Set}$ is $\mathbb{D}$-multicontinuous if for each diagram $S:\mathscr{D} \to \mathscr{A}^{\mathsf{op}}$ with $\mathscr{D} \in \mathbb{D}$ and with multicolimit $\alpha_i : S \to \Delta A_i$ $(i \in I)$, the evident cone

$$\coprod_{i\in I} F(A_i) \to FS(d)\ (d \in \mathsf{Obj}(\mathscr{D}))$$

in Set over FS is a limit. If $\mathscr{A}$ is a small category with $\mathbb{D}$-multicolimits, we denote by $\mathbb{D}\text{-}\mathsf{Multicont}(\mathscr{A}^{\mathsf{op}})$ the full subcategory of $[\mathscr{A}^{\mathsf{op}},\mathsf{Set}]$ consisting of the $\mathbb{D}$-multicontinuous functors.

Theorem 5.9. *For a category $\mathscr{K}$, the following conditions are equivalent*:
(i) $\mathscr{K}$ *is locally* $\mathbb{D}$-*multipresentable*;
(ii) $\mathscr{K}$ *is* $\mathbb{D}$-*accessible and has connected limits*;
(iii) $\mathscr{K}$ *is sketchable by a* ($\mathbb{D}$-*limit, coproduct*)-*sketch*;
(iv) $\mathscr{K} \simeq \mathbb{D}\text{-}\mathsf{Multicont}(\mathscr{A}^{\mathsf{op}})$ *for some small category* $\mathscr{A}$ *with* $\mathbb{D}$-*multicolimits.*

Proof. (i) $\Leftrightarrow$ (ii) by the results of Diers [11] which we have just recalled, because $\mathbb{D}$-accessible categories are accessible, see Corollary 4.4.

(i) $\Rightarrow$ (iv). We have $\mathscr{K} \simeq \mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K}_{\mathbb{D}})$. Since $\mathscr{K}_{\mathbb{D}}$ is closed under $\mathbb{D}$-multicolimits in $\mathscr{K}$, as proved in Remark 1.7, it is a $\mathbb{D}$-multicocomplete category. Now $\mathsf{Fam}(\mathscr{K}_{\mathbb{D}}^{\mathsf{op}})$ has $\mathbb{D}$-limits, since $\mathscr{K}_{\mathbb{D}}$ has $\mathbb{D}$-multicolimits, and a functor $F : \mathscr{K}_{\mathbb{D}}^{\mathsf{op}} \to \mathsf{Set}$ is $\mathbb{D}$-multicontinuous if and only if the essentially unique coproduct-preserving functor $G : \mathsf{Fam}(\mathscr{K}_{\mathbb{D}}^{\mathsf{op}}) \to \mathsf{Set}$ extending F is $\mathbb{D}$-continuous, and this is true if and only if $\mathrm{Lan}_Y F$ is $\mathbb{D}$-continuous. It follows that $\mathbb{D}\text{-}\mathsf{Multicont}(\mathscr{K}_{\mathbb{D}}^{\mathsf{op}}) \simeq \mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K}_{\mathbb{D}}) \simeq \mathscr{K}$.

(iv) $\Rightarrow$ (iii). We are going to present a sketch for $\mathbb{D}\text{-}\mathsf{Multicont}(\mathscr{A}^{\mathsf{op}})$ where $\mathscr{A}$ is a small, $\mathbb{D}$-multicocomplete category. Denote by α the supremum of the cardinals $\mathrm{card}\, I$, where I indexes a multicolimit of some $\mathbb{D}$-diagram in $\mathscr{A}$. Let $\mathscr{B}$ be a free completion of $\mathscr{A}^{\mathsf{op}}$ under coproducts of at most α objects. Then we have a sketch $\mathscr{S}$ on $\mathscr{B}$ whose $\mathbb{L}$-diagrams are all $\mathbb{D}$-diagrams in $\mathscr{A}$ to which σ assigns a multilimit (considered, of course, as a cone of D in $\mathscr{B}$) and whose $\mathbb{C}$-diagrams are all discrete diagrams of at most α objects of $\mathscr{A}^{\mathsf{op}}$, to which σ assigns the canonical coproduct cocone in $\mathscr{B}$. Then clearly

$$\mathsf{Mod}(\mathscr{S}) \cong \mathbb{D}\text{-}\mathsf{Multicont}(\mathscr{A}^{\mathsf{op}}).$$

(iii) $\Rightarrow$ (ii). If $\mathscr{S}$ is a ($\mathbb{D}$-limit, coproduct)-sketch on a category $\mathscr{A}$ then certainly $\mathsf{Mod}(\mathscr{S})$ is accessible; we must show that it is $\mathbb{D}$-accessible and has connected limits. Since coproducts commute in Set with connected limits, $\mathsf{Mod}(\mathscr{S})$ has connected limits.

Since $\mathsf{Mod}(\mathscr{S})$ is closed in $[\mathscr{A}, \mathsf{Set}]$ under $\mathbb{D}$-filtered colimits, it will suffice to find a small set $\mathscr{B}$ of $\mathbb{D}$-presentable objects in $\mathsf{Mod}(\mathscr{S})$ with the property that every object of $\mathsf{Mod}(\mathscr{S})$ is a $\mathbb{D}$-filtered colimit of objects of $\mathscr{B}$.

By [5, 4.32], $\mathsf{Mod}(\mathscr{S})$ is locally multipresentable; by [5, 2.45] the inclusion of $\mathsf{Mod}(\mathscr{S})$ in $[\mathscr{A}, \mathsf{Set}]$ satisfies the solution set condition; so by [5, 4.26], $\mathsf{Mod}(\mathscr{S})$ is multireflective in $[\mathscr{A}, \mathsf{Set}]$. For each A in $\mathscr{A}$, let $(r_{A,i} : \mathscr{A}(A, -) \to C_{A,i})_{i \in I_A}$ denote the multireflection into $\mathsf{Mod}(\mathscr{S})$ of the representable functor $\mathscr{A}(A, -)$. The functor

$$[\mathscr{A}, \mathsf{Set}]\ (\mathscr{A}(A, -), -) : [\mathscr{A}, \mathsf{Set}] \to \mathsf{Set}$$

preserves all colimits, so its restriction $E_A : \mathsf{Mod}(\mathscr{S}) \to \mathsf{Set}$ preserves $\mathbb{D}$-filtered colimits. Since

$$E_A \cong \coprod_{i \in I_A} \mathsf{Hom}(C_{A,i}, -),$$

the objects $C_{A,i}$ are $\mathbb{D}$-presentable; let $\mathscr{B}$ denote the closure under $\mathbb{D}^{\mathsf{op}}$-multicolimits in $\mathsf{Mod}(\mathscr{S})$ of the objects $C_{A,i}$ with $A \in \mathscr{A}$ and $i \in I_A$. By Remark 1.7, all objects in $\mathscr{B}$ are $\mathbb{D}$-presentable.

It remains to show that each $G:\mathscr{A} \to \mathsf{Set}$ in $\mathsf{Mod}(\mathscr{S})$ is a $\mathbb{D}$-filtered colimit of objects in $\mathscr{B}$. We know that G is the colimit of a diagram of shape $\mathscr{B} \downarrow G$; we shall show that $\mathscr{B} \downarrow G$ is $\mathbb{D}^{\mathsf{op}}$-cocomplete, hence, by Proposition 2.5, $\mathbb{D}$-filtered.

To do so, we write $P:\mathscr{B} \downarrow G \to \mathscr{B}$ for the canonical projection, and observe that $\mathscr{B} \downarrow G$ has a colimit of every diagram $S:\mathscr{C} \to \mathscr{B} \downarrow G$ for which PS has a multicolimit. Since $\mathscr{B}$ has multicolimits of $\mathbb{D}^{\mathsf{op}}$-diagrams, it follows that $\mathscr{B} \downarrow G$ is $\mathbb{D}^{\mathsf{op}}$-cocomplete. □

6. Distributivity of limits and colimits

Recall that all doctrines considered are assumed to be sound.

It follows from Proposition 3.3 that $\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K})$ is the object part of a pseudomonad $\mathbb{D}\text{-}\mathsf{Ind}$ on the quasi-category CAT of all categories. The definition of $\mathbb{D}\text{-}\mathsf{Ind}$ on morphisms (functors) follows from the universal property of free completions. Then $\eta_{\mathscr{K}}:\mathscr{K} \to \mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K})$ becomes a pseudonatural transformation. And the "multiplication"

$$\mu_{\mathscr{K}}:\mathbb{D}\text{-}\mathsf{Ind}(\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K})) \to \mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K})$$

is given by the essentially unique functor preserving $\mathbb{D}$-filtered colimits for which $\mu_{\mathscr{K}} \circ \eta_{\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K})} \cong \mathsf{id}$. In fact, each $\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K})$ is a KZ-pseudomonad, in the sense that $\mathbb{D}\text{-}\mathsf{Ind}(\eta_{\mathscr{K}}) \dashv \mu_{\mathscr{K}}$ for all categories $\mathscr{K}$; see [21].

In particular, for $\mathbb{D}=\emptyset$, we consider here the pseudomonad Colim of free completion under colimits (which can be identified with the above pseudomonad $\mathsf{Colim}(\mathscr{K})$ of all small presheaves). Dually we denote by Lim the pseudomonad on CAT of free completion under limits. $\mathsf{Lim}(\mathscr{K})$ can be identified with the subcategory of $[\mathscr{K},\mathsf{Set}]^{\mathsf{op}}$ formed by all small functors.

Remark 6.1. The pseudomonad $\mathbb{D}\text{-}\mathsf{Ind}$ obviously preserves all ∞-*filtered bicolimits* in CAT; that is, all large bicolimits whose scheme is a category $\mathscr{D}$ which is λ-filtered for all cardinals λ.

Remark 6.2. In the next theorem we prove a fact about $\mathbb{D}$-filtered colimits which seems to be new even for $\mathbb{D}=\emptyset$ (where every colimit is $\mathbb{D}$-filtered). So let us comment on this most general case first. We state that

free colimits distribute over free limits.

That is, the pseudomonads

Lim, of free completion under limits

and

Colim, of free completion under colimits

in the quasi-category CAT admit a distributive law $\mathsf{LimColim} \to \mathsf{ColimLim}$. For this, we only need to prove that Colim lifts from CAT to the quasi-category of all Lim-algebras (i.e., complete categories and continuous functors); see [22]. That is, we need to prove that for $\mathscr{K}$ complete, also $\mathsf{Colim}(\mathscr{K})$ is complete, and for $H:\mathscr{K} \to \mathscr{L}$ continuous, also $\mathsf{Colim}(H)$ is continuous. The former can be derived from results of Freyd [13], but we present a simpler direct proof. The method of the proof below has already been used in [3] for $\mathbb{D} = \mathsf{FINPR}$.

Theorem 6.3. ($\mathbb{D}$-filtered colimits distribute over limits).
(i) *If $\mathscr{K}$ is complete, then $\mathbb{D}$-$\mathsf{Ind}(\mathscr{K})$ is complete*;
(ii) *if $\mathscr{K}$ and $\mathscr{L}$ are complete, and $H:\mathscr{K} \to \mathscr{L}$ is continuous, then $\mathbb{D}$-$\mathsf{Ind}(H)$ is continuous*;
(iii) *if $\mathscr{K}$ is complete, then both $\eta_{\mathscr{K}}$ and $\mu_{\mathscr{K}}$ are continuous.*

Proof. We only need to prove (i) and (ii), since (iii) will follow: $\mu_{\mathscr{K}}$ will be continuous because it is a right adjoint, while $\eta_{\mathscr{K}}$ will be continuous since it is a codomain restriction of the Yoneda embedding and $\mathbb{D}$-$\mathsf{Ind}(\mathscr{K})$ is closed in $[\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ under limits; this last fact will follow by the Yoneda lemma from the completeness of $\mathbb{D}$-$\mathsf{Ind}(\mathscr{K})$, since $\mathbb{D}$-$\mathsf{Ind}(\mathscr{K})$ is a full subcategory of $[\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ containing the representables.

(I) We first prove (i) and (ii) for the special case that

$$\mathscr{K} = \mathsf{Lim}(\mathscr{K}_0), \quad \mathscr{L} = \mathsf{Lim}(\mathscr{L}_0)$$

for some categories $\mathscr{K}_0$ and $\mathscr{L}_0$ and

$$H = \mathsf{Lim}(H_0) \text{ for some } H_0:\mathscr{K}_0 \to \mathscr{L}_0.$$

(Ia) Let $\mathscr{K}_0$ and $\mathscr{L}_0$ be small. Then $\mathsf{Lim}(\mathscr{K}_0)$ is dual to $[\mathscr{K}_0, \mathsf{Set}]$, a locally presentable category. Consequently, a functor $F:(\mathsf{Lim}(\mathscr{K}_0))^{\mathsf{op}} \to \mathsf{Set}$ is small if and only if it is accessible; while by Remark 2.10, it is $\mathbb{D}$-flat if and only it is small and $\mathbb{D}$-continuous. Limits of accessible functors are accessible [20], and limits of $\mathbb{D}$-continuous functors are $\mathbb{D}$-continuous. It follows that a limit of $\mathbb{D}$-flat functors in $[\mathscr{K}^{\mathsf{op}}, \mathsf{Set}]$ is $\mathbb{D}$-flat. This proves (i).

The functor

$$\mathsf{Lim}(H_0):[\mathscr{K}_0, \mathsf{Set}]^{\mathsf{op}} \to [\mathscr{L}_0, \mathsf{Set}]^{\mathsf{op}}$$

is obviously right adjoint to the functor

$$(H_0, -)^{\mathsf{op}}:[\mathscr{L}_0, \mathsf{Set}]^{\mathsf{op}} \to [\mathscr{K}_0, \mathsf{Set}]^{\mathsf{op}}.$$

Consequently, $\mathsf{Colim}(H) = \mathsf{Colim}(\mathsf{Lim}\, H_0)$ is also a right adjoint, which proves that it is continuous.

(Ib) Let $\mathscr{K}_0$ and $\mathscr{L}_0$ be arbitrary. Express $\mathscr{K}_0$ as an ∞-filtered bicolimit of small subcategories $\mathscr{K}_t$, $t \in T$, and denote by $E_{t,t'}:\mathscr{K}_t \to \mathscr{K}_{t'}$ the connecting functors, for $\mathscr{K}_t \subseteq \mathscr{K}_{t'}$. By (Ia), the $\mathbb{D}$-$\mathsf{Ind}(\mathsf{Lim}\, E_{t,t'})$ are continuous functors between complete categories. Since $\mathbb{D}$-Ind and Lim preserve ∞-filtered bicolimits, we see that

$$\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K}) \simeq \operatorname*{bicolim}_{t \in T} \mathbb{D}\text{-}\mathsf{Ind}(\mathsf{Lim}\mathscr{K}_t)$$

is an ∞-filtered bicolimit of complete categories with continuous connecting functors—it follows immediately that $\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K})$ is complete. This proves (i).

The proof of (ii) is analogous. Express $\mathscr{K}_0$ and $\mathscr{L}_0$ as ∞-filtered bicolimits of small subcategories $\mathscr{K}_t \subseteq \mathscr{K}_0$ and $\mathscr{L}_t \subseteq \mathscr{L}_0$ in such a way that $H(\mathscr{K}_t) \subseteq \mathscr{L}_t$ for each $t \in T$. For the domain–codomain restrictions $H_t : \mathscr{K}_t \to \mathscr{L}_t$ we then clearly have $H_0 \simeq \mathrm{bicolim}_{t\in T} H_t$ and we conclude

$$\mathbb{D}\text{-}\mathsf{Ind}(H) \simeq \operatorname*{bicolim}_{t\in T} \mathbb{D}\text{-}\mathsf{Ind}(\mathsf{Lim}(H_t)).$$

By (Ia), $\mathbb{D}\text{-}\mathsf{Ind}(H)$ is thus the ∞-filtered bicolimit of the continuous functors $\mathbb{D}\text{-}\mathsf{Ind}(\mathsf{Lim}\, H_t)$ with continuous connecting functors—it follows immediately that $\mathbb{D}\text{-}\mathsf{Ind}(H)$ is continuous.

(II) We prove (i) in general. If $\mathscr{K}$ is complete then there is an essentially unique continuous functor $L_{\mathscr{K}} : \mathsf{Lim}\mathscr{K} \to \mathscr{K}$ with

$$L_{\mathscr{K}} \circ \eta_{\mathscr{K}}^{\mathsf{Lim}} \cong \mathsf{id} \quad \text{and} \quad \eta_{\mathscr{K}}^{\mathsf{Lim}} \dashv L_{\mathscr{K}}.$$

Thus $L_{\mathscr{K}}$ is a right adjoint left inverse of $N_{\mathscr{K}} = \eta_{\mathscr{K}}^{\mathsf{Lim}}$, and so $\mathbb{D}\text{-}\mathsf{Ind}(L_{\mathscr{K}})$ is right adjoint left inverse of $\mathbb{D}\text{-}\mathsf{Ind}(N_{\mathscr{K}})$. We know from (I) that $\mathbb{D}\text{-}\mathsf{Ind}(\mathsf{Lim}\,\mathscr{K})$ is complete; it follows that $\mathbb{D}\text{-}\mathsf{Ind}(\mathscr{K})$ is complete. Furthermore, we have the formula:

$$\lim S \cong \mathbb{D}\text{-}\mathsf{Ind}(L_{\mathscr{K}})\lim(\mathbb{D}\text{-}\mathsf{Ind}(N_{\mathscr{K}})S).$$

(III) We prove (ii) in general. Use the notation $L_{\mathscr{L}}$ and $N_{\mathscr{L}}$ as in (II) and observe that the continuity of $H : \mathscr{K} \to \mathscr{L}$ is equivalent to stating that the square

$$\begin{tikzcd}
\mathsf{Lim}(\mathscr{K}) \arrow[r, "\mathsf{Lim}(H)"] \arrow[d, "L_{\mathscr{K}}"'] & \mathsf{Lim}(\mathscr{L}) \arrow[d, "L_{\mathscr{L}}"] \\
\mathscr{K} \arrow[r, "H"'] & \mathscr{L}
\end{tikzcd}$$

commutes up to a natural isomorphism. By (*I*), $\mathbb{D}\text{-}\mathsf{Ind}(\mathsf{Lim}\, H)$ is continuous. Using the formula for lim S in (II) we establish the continuity of $\mathbb{D}\text{-}\mathsf{Ind}(H)$:

$$\begin{aligned}
\mathbb{D}\text{-}\mathsf{Ind}(H)(\lim S) &\cong \mathbb{D}\text{-}\mathsf{Ind}(H)\mathbb{D}\text{-}\mathsf{Ind}(L_{\mathscr{K}})\lim(\mathbb{D}\text{-}\mathsf{Ind}(N_{\mathscr{K}})S) \\
&\cong \mathbb{D}\text{-}\mathsf{Ind}(L_{\mathscr{L}})\mathbb{D}\text{-}\mathsf{Ind}(\mathsf{Lim}\, H)\lim(\mathbb{D}\text{-}\mathsf{Ind}(N_{\mathscr{K}})S) \\
&\cong \mathbb{D}\text{-}\mathsf{Ind}(L_{\mathscr{L}})\lim(\mathbb{D}\text{-}\mathsf{Ind}(\mathsf{Lim}\, H)\mathbb{D}\text{-}\mathsf{Ind}(N_{\mathscr{K}})S) \\
&\cong \mathbb{D}\text{-}\mathsf{Ind}(L_{\mathscr{L}})\lim(\mathbb{D}\text{-}\mathsf{Ind}(N_{\mathscr{L}})\mathbb{D}\text{-}\mathsf{Ind}(H)S) \\
&\cong \lim(\mathbb{D}\text{-}\mathsf{Ind}(H)S),
\end{aligned}$$

where the first and the last isomorphisms hold by (II) and the third one by (I). $\square$

Corollary 6.4. $\mathbb{D}$-Ind *distributes over* Lim *for all sound doctrines* $\mathbb{D}$.

In fact, the quasi-category of algebras of Lim is precisely the quasi-category of all complete categories and continuous functors. Thus, Theorem 6.3 gives a lifting of the pseudomonad $\mathbb{D}$-Ind from CAT to Alg(Lim), and this is equivalent to specifying a distributive law of $\mathbb{D}$-Ind over Lim. Moreover, the distributive law is unique since Lim is a co-KZ-pseudomonad: see [21].

Remark 6.5. (i) We thus see, as announced above, that

Colim *distributes over* Lim.

(ii) The distributive law above allows us to form the composite pseudomonad $(\mathbb{D}\text{-Ind})\circ\text{Lim}$. It follows from [23] that algebras of that pseudomonad form a 2-category equivalent to the following one:

0-cells (the objects) are the categories $\mathscr{K}$ with limits and $\mathbb{D}$-filtered colimits, such that the colimit functor colim: $\mathbb{D}\text{-Ind}(\mathscr{K}) \to \mathscr{K}$ is continuous;

1-cells (the morphisms) are the functors preserving limits and $\mathbb{D}$-filtered colimits;

2-cells are the natural transformations.

For $\mathbb{D}=$FIN these 0-cells are called *precontinuous categories* [2], and for $\mathbb{D}=$FINPR they are called *algebraically exact categories* [3].

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